Tutorial 1 (Jan 14, 16)

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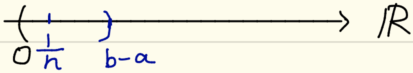
Q0) (Supplementary lemma to Lecture 2, eg. 5)

Show that the set of rational numbers $\mathbb{Q}$ is dense in $\mathbb{R}$:

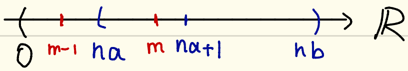
for any open interval $(a,b) \subseteq \mathbb{R}$, we have $\mathbb{Q} \cap (a,b) \neq \phi$.

Sol: WLOG assume $a > 0$. Then $\exists\, n \in \mathbb{N}$ such that

$$0 < \frac{1}{n} < b-a$$



$$\Rightarrow 0 < na < na+1 < nb$$



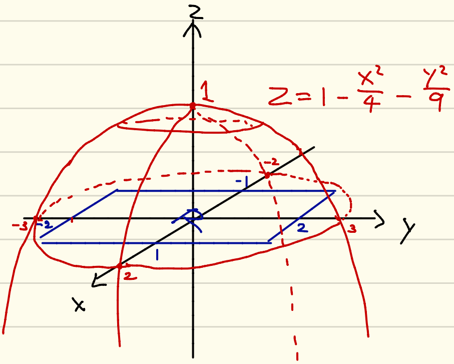
Choose $m \in \mathbb{N}$ such that $m-1 \leqslant na < m$

$\therefore\ na < m < nb \quad \Rightarrow \quad a < \frac{m}{n} < b. \quad \therefore\ \frac{m}{n} \in \mathbb{Q} \cap (a,b)$

Q1) Find the volume of the region bounded above by the elliptic paraboloid

$z = 1 - \frac{x^2}{4} - \frac{y^2}{9}$ and below by the rectangle $R = [-1,1] \times [-2,2]$

Picture:



Sol: Define $f: R \to \mathbb{R}$ by $f(x,y) = 1 - \frac{x^2}{4} - \frac{y^2}{9}$, then

$$\text{Volume} \triangleq \iint_R f(x,y)\,dA = \int_{-1}^{1}\int_{-2}^{2} \left(1 - \frac{x^2}{4} - \frac{y^2}{9}\right) dy\,dx \quad \text{(by Fubini's Theorem)}$$

$$= 4\int_0^1\int_0^2 \left(1 - \frac{x^2}{4} - \frac{y^2}{9}\right) dy\,dx \quad \left(\text{as } f(x,y) \text{ is an even function with respect to } x, y\right)$$

$$= 4\int_0^1 \left[\left(1 - \frac{x^2}{4}\right)y - \frac{y^3}{27}\right]_0^2 dx$$

$$= 4\int_0^1 \left(2\left(1 - \frac{x^2}{4}\right) - \frac{8}{27}\right) dx$$

$$= 4\int_0^1 \left(-\frac{x^2}{2} + \frac{46}{27}\right) dx$$

$$= 4 \cdot \left[-\frac{x^3}{6} + \frac{46}{27}x\right]_0^1 = 4 \cdot \left(-\frac{1}{6} + \frac{46}{27}\right) = \frac{166}{27}$$

Q2) Evaluate $\iint_R f(x,y)\,dA$, where $R = [0,1] \times [-3,3]$; $f(x,y) = \frac{xy^2}{x^2+1}$

Sol Method 1: Using Fubini's Theorem and compute the iterated integral.

$$\iint_R f(x,y)\,dA = \int_0^1 \int_{-3}^{3} \frac{xy^2}{x^2+1}\,dy\,dx$$
$$= \int_0^1 \left[\left(\frac{x}{x^2+1}\right)\left(\frac{y^3}{3}\right) \right]_{-3}^{3} dx$$
$$= \int_0^1 \frac{x}{x^2+1} \cdot 18\,dx$$
$$= 9 \cdot \int_0^1 \frac{d(x^2+1)}{x^2+1} = 9 \cdot \left[\log(x^2+1)\right]_0^1 = 9 \log 2$$

Method 2: Using Fubini's Theorem with a separation of variables of $f(x,y)$.

i.e. $f(x,y) = g(x)h(y)$, where $\begin{cases} g(x) = \frac{x}{x^2+1} : [0,1] \to \mathbb{R} \\ h(y) = y^2 : [-3,3] \to \mathbb{R} \end{cases}$

then $$\iint_R f(x,y)\,dA = \int_0^1 \int_{-3}^{3} g(x)h(y)\,dy\,dx$$
$$= \int_0^1 g(x) \left(\int_{-3}^{3} h(y)\,dy \right) dx$$
$$= \left(\int_0^1 g(x)\,dx \right) \left(\int_{-3}^{3} h(y)\,dy \right) = \left(\tfrac{1}{2}\log 2\right) \cdot (18) = 9 \log 2$$

Q3) Evaluate $\int_0^2 \int_0^3 y e^{-xy} \, dy \, dx$.

Sol First attempt: Directly compute the iterated integral.

$$\int_0^2 \int_0^3 y e^{-xy} \, dy \, dx = \int_0^2 \left(\int_0^3 y \cdot \underbrace{\left(-\frac{1}{x}\right)}_{\text{NOT defined at } x=0!} d\left(e^{-xy}\right) \right) dx$$

X

Correct attempt: Using Fubini's Theorem to interchange the order of integration.

$$\begin{aligned}
\int_0^2 \int_0^3 y e^{-xy} \, dy \, dx &= \int_0^3 \int_0^2 y e^{-xy} \, dx \, dy \quad \text{(By Fubini's Thm)} \\
&= \int_0^3 \left(\int_0^2 e^{-xy} \, d(xy) \right) dy \\
&= \int_0^3 \left[-e^{-xy} \right]_0^2 dy \\
&= \int_0^3 \left(-e^{-2y} + 1 \right) dy \\
&= \left[\frac{e^{-2y}}{2} + y \right]_0^3 \\
&= \left(\frac{e^{-6}}{2} + 3 \right) - \frac{1}{2} = \frac{e^{-6} + 5}{2}
\end{aligned}$$